Consider the following system:

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} v_{1,k} \\ v_{2,k} \end{bmatrix} \tag{1}$$

$$y_k = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} \tag{2}$$

where

$$\begin{aligned} v_{1,k} &\sim N(0, V_1) \\ v_{2,k} &\sim N(0, V_2) \end{aligned}$$

equation (1) can be re-written as

$$x_{1,k+1} = A_{11}x_{1,k} + A_{12}x_{2,k} + v_{1,k} \tag{3}$$

$$x_{2,k+1} = A_{21}x_{1,k} + A_{22}x_{2,k} + v_{2,k} \tag{4}$$

the reduced order state estimate $\hat{x}_{2,k+1}$ can be written as

$$\begin{aligned} \hat{x}_{2,k+1} &= A_{21}x_{1,k} + A_{22}\hat{x}_{2,k} + K(x_{1,k+1} - A_{11}x_{1,k} - A_{12}\hat{x}_{2,k}) \\ &= (A_{22} - KA_{12})\hat{x}_{2,k} + (A_{21} - KA_{11})x_{1,k} + Kx_{1,k+1} \end{aligned} \tag{5}$$

based on equation (4) and (5), estimation error $e_{x_2,k+1}$ can be written as

$$\begin{aligned} e_{x_2,k+1} &= x_{2,k+1} - \hat{x}_{2,k+1} \\ &= A_{21}x_{1,k} + A_{22}x_{2,k} + v_{2,k} - (A_{22} - KA_{12})\hat{x}_{2,k} - (A_{21} - KA_{11})x_{1,k} - Kx_{1,k+1} \\ &= KA_{11}x_{1,k} - Kx_{1,k+1} + A_{22}x_2 + v_{2,k} - (A_{22} - KA_{12})\hat{x}_{2,k} \end{aligned} \tag{6}$$

substitute equation (3) into equation (6)

$$\begin{aligned} e_{x_2,k+1} &= KA_{11}x_{1,k} - K(A_{11}x_{1,k} + A_{12}x_{2,k} + v_{1,k}) + A_{22}x_2 + v_{2,k} - (A_{22} - KA_{12})\hat{x}_{2,k} \\ &= -KA_{12}x_{2,k} - Kv_{1,k} + A_{22}x_2 + v_{2,k} - (A_{22} - KA_{12})\hat{x}_{2,k} \\ &= (A_{22} - KA_{12})x_{2,k} - (A_{22} - KA_{12})\hat{x}_{2,k} - Kv_{1,k} + v_{2,k} \\ &= (A_{22} - KA_{12})e_{x_2,k} - Kv_{1,k} + v_{2,k} \end{aligned} \tag{7}$$

x_2's estimation error covariance can be found using the Riccati equation

$$P_{k+1} = (A_{22} - KA_{12})P_k(A_{22} - KA_{12})^T + V_2 + KV_1K^T \tag{8}$$

in the form

$$P_{k+1} = (A - KC)^T P_k(A - KC) + V + KWK^T \tag{9}$$

so

$$K_k = AP_kC^T(CP_kC^T + W)^{-1} \tag{10}$$

in this csae

$$K_k = A_{22} P_k A_{12}^T (A_{12} P_k A_{12}^T + V_1)^{-1} \tag{11}$$

Practical consideration: since we don't know what $x_{1,k+1}$ is at time k, instead of solving $\hat{x}_{2,k}$, we introduce $\hat{\mathscr{X}}_k = \hat{x}_{2,k} - K x_{1,k}$

in this csae

$$\begin{aligned} \hat{\mathscr{X}}_{k+1} &= \hat{x}_{2,k+1} - K x_{1,k+1} \\ &= (A_{22} - K A_{12}) \hat{x}_{2,k} + (A_{21} - K A_{11}) x_{1,k} + K x_{1,k+1} - K x_{1,k+1} \\ &= (A_{22} - K A_{12}) \hat{x}_{2,k} + (A_{21} - K A_{11}) x_{1,k} \\ &= (A_{22} - K A_{12}) \hat{\mathscr{X}}_k + [(A_{22} - K A_{12}) K + A_{21} - K A_{11}] x_{1,k} \end{aligned} \tag{12}$$

after solving $\hat{\mathscr{X}}_k$, $\hat{x}_{2,k}$ can be found from

$$\hat{x}_{2,k} = \hat{\mathscr{X}}_k + K x_{1,k} \tag{13}$$

In terms of the steady-state solution, we can find the constant Kalman gain by solving Algebric Riccati equation. For the nearly constant velocity (NCV) model, since we have $A_{11} = D_1$ (D means diagonal), $A_{12} = I$, $A_{21} = 0$, $A_{22} = D_2$, the Algebric Riccati equation can be represented as

$$P = D_2 P D_2^T - D_2 P (P + V_1)^{-1} P D_2^T + V_2 \tag{14}$$

Constant Kalman gain is

$$K = D_2 P (P + V_1)^{-1} \tag{15}$$

state estimation equation is

$$\hat{\mathscr{X}}_{k+1} = (D_2 - K) \hat{\mathscr{X}}_k + [(D_2 - K) K - K D_1] x_{1,k} \tag{16}$$

Finally, $\hat{x}_{2,k}$ can be found from

$$\hat{x}_{2,k} = \hat{\mathscr{X}}_k + K x_{1,k} \tag{17}$$

```
clear all
close all
clc

maxT=2000; % max sim time

x = zeros(4,maxT); % save results

% initial position, velocity
x(:,1) = [0;
          5;
          0;
          7];


T = 1; % sampling time


A = [0.95 0    T  0; %x -> position information
     0    0.95 0  T; %y
     0    0    1  0; %x_dot
     0    0    0  1];%y_dot

A11 = A(1:2,1:2);
A12 = A(1:2,3:4);
A21 = A(3:4,1:2);
A22 = A(3:4,3:4);

% F = [T^2/2 0;
%      T^2/2 0;
%      0     T;
%      0     T];

C = [1 0 0 0;
     0 1 0 0];

G = [1 0;
     0 1];

% state noise covariance matrix
V1 = [0.85 0;
      0    0.3];

V2 = [0.4 0;
```

```
      0   0.6];


% % measurement noise covariance matrix
% W = [0.28 0;
%       0    0.65];


for k = 1:maxT, % simulate model
    x(:,k+1) = A*x(:,k) + [V1*randn(2,1); V2*randn(2,1)];
    y(:,k) = C*x(:,k);
end


t = 1:maxT+1;

% figure,
% plot(x(1,:),x(2,:), 'Linewidth', 2)
% title('x1 vs x2')

% figure,
% plot(t, x(3,:),'Linewidth', 2)
% title('speed, x3')
%
% figure,
% plot(t,x(4,:),'Linewidth', 2)
% title('speed, x4')

%%
% reduced order Kf
P(:,:,1) = 10*eye(2);
S_x_hat(:,1) = [1; 2];

for k = 1:maxT, % simulate model
    K = A22*P(:,:,k)*A12'*inv(A12*P(:,:,k)*A12' + V1);
    S_x_hat(:,k+1) = (A22 - K*A12)*S_x_hat(:,k) + ((A22 - K*A12
       )*K + A21 - K*A11)*x(1:2,k);
    P(:,:,k+1) = (A22 - K*A12)*P(:,:,k)*(A22 - K*A12)' + V2 + K
       *V1*K';
    x2_hat(:,k+1) = S_x_hat(:,k+1) + K*x(1:2,k);
end


figure,
plot(t, x(3,:),'Linewidth', 2)
```

```
hold on
plot(t, x2_hat(1,:),':','Linewidth', 2)
title('x3 vs x3 hat, reduced order Kf')

figure,
plot(t,x(4,:),'Linewidth', 2)
hold on
plot(t, x2_hat(2,:),':','Linewidth', 2)
title('x4 vs x4 hat, reduced order Kf')


%%
% full order Kf

P_full(:,:,1) = 10*eye(4);
x_hat(:,1) = [1; 2; 3; 4];

for k = 1:maxT, % simulate model
    K_full = A*P_full(:,:,k)*C'*inv(C*P_full(:,:,k)*C');
    x_hat(:,k+1) = A*x_hat(:,k) + K_full*(y(:,k) - C*x_hat(:,k)
       );
    P_full(:,:,k+1) = A*P_full(:,:,k)*A' + blkdiag(V1,V2) -
       K_full*C*P_full(:,:,k)*A';
end


% figure,
% plot(t, x(1,:),'Linewidth', 2)
% hold on
% plot(t, x_hat(1,:),':','Linewidth', 2)
% legend('x1','x1 hat')
% title('x1 vs x1 hat, full order Kf')
%
% figure,
% plot(t,x(2,:),'Linewidth', 2)
% hold on
% plot(t, x_hat(2,:),':','Linewidth', 2)
% legend('x2','x2 hat')
% title('x2 vs x2 hat, full order Kf')


figure,
plot(x(1,:),x(2,:),'Linewidth', 2)
hold on
plot(x_hat(1,:), x_hat(2,:),':','Linewidth', 2)
```

```
legend('traj','traj hat')
title('traj vs traj hat, full order Kf')


figure,
plot(t,x(3,:),'Linewidth', 2)
hold on
plot(t, x_hat(3,:),':','Linewidth', 2)
legend('x3','x3 hat')
title('x3 vs x3 hat, full order Kf')

figure,
plot(t,x(4,:),'Linewidth', 2)
hold on
plot(t, x_hat(4,:),':','Linewidth', 2)
legend('x4','x4 hat')
title('x4 vs x4 hat, full order Kf')

error_reduced_x3 = abs(x(3,:) - x2_hat(1,:));
error_full_x3 = abs(x(3,:) - x_hat(3,:));

error_reduced_x4 = abs(x(4,:) - x2_hat(2,:));
error_full_x4 = abs(x(4,:) - x_hat(4,:));

figure,
plot(t,error_reduced_x3,'Linewidth', 2)
hold on
plot(t, error_full_x3,':','Linewidth', 2)
legend('x3 reduced order error','x3 full order error')
title('x3 reduced vs x3 full, mean square error')

figure,
plot(t,error_reduced_x4,'Linewidth', 2)
hold on
plot(t, error_full_x4,':','Linewidth', 2)
legend('x4 reduced order error','x4 full order error')
title('x4 reduced vs x4 full, mean square error')
```